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**THERMAL RADIATION CHARACTERISTICS OF TRANSPARENT,  
SEMI-TRANSPARENT AND TRANSLUCENT MATERIALS  
UNDER NON-ISOTHERMAL CONDITIONS**

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Robert C. Folweiler)


## ABSTRACT

✓ Thermal radiation from refractory oxide materials is a volume, rather than a surface, phenomenon. General equations describing emissivity, transmissivity, and reflectivity for scattering and absorbing systems have been derived in a form that allows determination of the necessary material constants and calculation of emissivity under isothermal and non-isothermal conditions. Methods of evaluating conditions under which these equations are applicable have been derived, and a simple expression developed for radiant energy transfer when conduction in the solid is negligible relative to the radiant energy transfer in the solid.

Experimental apparatus has been developed and measurements of the isothermal emissivity of several refractory oxides and the transmissivity of single crystals have been made over the wavelength range 1-15 microns at temperatures up to 1350°C. Emissivity was found to depend strongly on composition, microstructure and wavelength. Different grades of "alumina" ceramics range from about 0.2 to above 0.9 in the near infrared (~~1-4~~).

Diffuse transmissivity measurements provided experimental values from which scattering was calculated for each sample. Large scattering values (~~100-1000 cm<sup>-1</sup>~~) were obtained, however, emissivities calculated from these measurements agreed well with experimentally measured data. Results show that the normal emissivity in these systems is substantially larger than the hemispherical emissivity in contrast to opaque solids where they are nearly the same.

This technical documentary report has been reviewed and is approved.

  
JULES I. WITTEBORT  
Chief, Thermal and Solid  
State Branch  
Materials Physics Division  
Air Force Materials Laboratory

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# LIST OF SYMBOLS

A	Constant in Hamaker equation
a	Absorption coefficient of material with scattering centers
$\alpha$	Absorption coefficient for single crystal; also absorbance of a solid layer
b	Radiation constant, $b = 4\sigma'n^2T^3$
$b_1$	Radiation constant for $n = 1$ , $b_1 = 4\sigma'T^3$
B	Constant in Hamaker equation
$\beta$	Material constant, $\beta = \sigma/(a+2s)$
$\beta_0$	Material constant, $\beta_0 = \sqrt{a/(a+2s)} = c_0/(a+2s)$
$\gamma$	Angle of refraction
D	Sample thickness
$D_p$	Pore thickness
$\epsilon$	Emissivity
E	Blackbody radiant energy flux
$E_0$	Blackbody radiant energy flux at $x = 0$
$E_H$	Hamaker approximation to blackbody radiant energy flux
$\eta$	Energy gradient at surface: $(\frac{dE}{dx})$ surface
$\xi$	Variable of integration ranging from $T_0$ to $\theta$ , with dimensions of temperature
$f_1$	defined functions
:	
:	
$f_6$	
H	Net heat flux
I	Radiant energy flux in the direction of the positive x axis
$I_i$	Incident radiant energy flux at $x = 0$
$I_0$	Forward radiant energy flux in solid at $x = 0$

$I_D$	Forward radiant energy flux in solid at $x = D$
$I_e$	Radiant energy flux in positive $x$ direction at a surface
$I_H$	Hamaker approximation to radiant energy flux in positive $x$ direction
$I_h$	Initial radiant energy flux passing through diffusing screen
$I_x$	Forward radiant energy flux in solid $x = x$
$J$	Radiant energy flux in the direction of the negative $x$ axis
$J_0$	Backward radiant energy flux in solid at $x = 0$
$J_D$	Backward radiant energy flux in solid at $x = D$
$J_e$	Radiant energy flux in negative $x$ direction at a surface
$J_H$	Hamaker approximation to radiant energy flux in negative $x$ direction
$J_x$	Backward radiant energy flux in solid at $x = x$
$k$	Lattice thermal conductivity
$k_e$	Effective conductivity, $k_e = -Q'x/\Delta T$
$\kappa$	Material constant in Hamaker equation, $\kappa = 2b/k(a+2s)$
$\lambda$	Wavelength
$P$	Volume fraction porosity
$Q$	Heat flow across a unit area
$Q_{-g}$	Negative of heat removed from surface by conductivity
$n$	Index of refraction
$r_e$	Reflectivity of ray incident at angle $\theta$
$r$	Reflectivity of absorbing and scattering solid layer
$r_i$	Reflectivity of radiant energy emergent from solid
$r_o$	Reflectivity of radiant energy incident on solid
$r_s$	Reflectivity of diffusing screen

LIST OF SYMBOLS, continued

- s Scattering coefficient
- $\sigma$  Material constant for non-isothermal case,  $\sigma = \sigma_0 (1+\epsilon)^{\frac{1}{2}}$
- $\sigma_0$  Material constant,  $\sigma_0 = \sqrt{a(a+2s)}$
- $\sigma'$  Stefan-Boltzmann radiation constant
- T Temperature, absolute scale
- $T_0$  Average temperature, absolute scale
- $\tau$  Radiant energy transmissivity of absorbing and scattering solid layer
- $\theta$  Angle of incidence
- $\Theta$  Variable of integration ranging from  $T_0$  to T with dimensions of temperature
- x Linear dimension in direction of heat flow

## 1.0 INTRODUCTION

Performance evaluation of ceramic, ceramic-plastic or ceramic-metal fiber radomes, re-entry bodies or high temperature anti-abrasion coatings on a metal substrate requires a precise knowledge of the thermal gradients and radiant thermal transfer through and from the bodies. Energy transfer through these systems is by two modes: (1) radiation, (2) lattice conductivity. There exist complex interactions and perturbations between these transfer modes which cause large deviations from linear thermal gradients in areas near interfaces. These gradients may be larger or smaller than would be predicted on the basis of lattice conduction alone.

Emissivity has been widely accepted as a simple dimensionless constant, and has been used in conjunction with the blackbody radiation equation to uniquely express the quantity and quality of radiant energy emitted by a heated object. For opaque materials such as metals or graphite which possess large extinction coefficients, the emission process is in essence a surface phenomena, unaffected by internal gradients, and the emissivity concept for these systems is accurate and useful. However, for semi-transparent materials such as polycrystalline ceramics used for high temperature applications, radiant energy emission is a volume process. Thermal gradients normal to the radiating surface are almost always present, and no obvious single temperature is useful in characterizing such radiating systems. Thus, the surface emissivity concept becomes fundamentally unsound.

The importance and significance of these concepts to heat-transfer and temperature measurement in translucent materials such as refractory oxides and oxide coatings is not widely appreciated. However, using reasonable optical constants for dense sintered aluminum oxide, our estimates indicate that differences as great as one hundred percent from isothermal emissivity values may occur in practical systems. For example, at 1800 C the effective emissivity

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for aluminum oxide in still air is already thirty-five percent higher than the isothermal value. This difference becomes increasingly important as the temperature level increases (roughly proportional to  $T$ ), and as the air velocity increases. That is, these effects are substantial ones of practical as well as theoretical importance.

McMahon<sup>(1)</sup>, Gardon<sup>(2)</sup>, and others<sup>(3,4,5,6,7)</sup> have developed the volume emissivity concept for describing the radiant transfer within and from transparent (i.e. non-scattering) systems such as glass or single crystals. However, most important ceramic and plastic systems contain either pores or fibers as scattering centers. As a result, these analytical methods and use of a simple volume emissivity are unsound. There is not at present any sound general basis for describing the thermal radiation characteristics of semi-transparent materials under non-isothermal conditions.

The numerical method of Hottel<sup>(8)</sup> and the Hamaker<sup>(9)</sup> differential equations consider both absorption and scattering coefficients. At present neither of these systems has been subjected to much high temperature experimental verification. Thus it is necessary to (a) select an analytical system sufficiently powerful to solve complex and difficult problems and which will permit material constants to be expressed in a relatively simple form, and (b) to experimentally verify the analytical results.

## 2.0 THE PROBLEM

The problem of heat transfer and radiant energy interaction with absorbing and scattering materials is a complex one. Factors such as absorption and reemission within the body as well as scattering must be taken into account; in addition to the linear thermal gradients in the center of thick sections, boundary effects must be considered; also there are dimensional effects and, less frequently, polarization and phase have to be handled. In general there are two methods of approach to this problem.

Representative of one, the numerical approach, is the Hottel-Cohen method<sup>(8)</sup>. In this method the sample being considered is broken up into small zones of constant temperature. Radiation exchange factors are determined for each possible zone interchange. These radiation exchange factors are then used in energy balance equations, one equation for each zone in the system. Heat transfer to the zone by convection, radiation or by unsteady state enthalpy change of the zone are considered in the energy balance equations. When these are solved with suitable boundary conditions, the required temperature and heat flux distribution inside the sample are determined. The success of this type of calculation depends on the speed of modern computers. One of the advantages is that (by making the zones small enough) practically any desirable accuracy can be obtained, provided that sufficient machine time is available. A main disadvantage is that a solution is valid for only one specific case; this is useful when actual heat transfer to some known configuration with known optical properties is desired, but is not helpful for general design purposes, or in understanding the mechanisms involved. It does not provide a good basis for defining material properties or comparing the potential utility of different kinds of materials.

The other general method involves equations describing the whole process in terms of position and the parameters at these positions. The solution describes the desired property, such as effective thermal conductivity, in terms of the necessary parameters. The main disadvantage of this technique is that it is extremely complicated to solve the problem in the most general case, and the solution may be so complicated that a computer is needed to obtain numerical values from it. A practical approach is to make suitable approximations which simplify the general problem to a manageable set of equations. Such an approach was taken by Hamaker<sup>(9)</sup> who assumed diffuse radiation and divided the flux into a forward and backward component. Though the limitation to diffuse radiation simplifies the mathematics a great deal, it is not a serious loss of generality since practically all cases of heat transfer of practical interest involve mostly diffuse radiation.

Although other analytical methods are possible, Hamaker's equations appear to be the most useful system presently available to characterize the thermal radiation behavior of both isothermal and non-isothermal semi-transparent solids. For non-isothermal cases, thermal gradients as well as optical coefficients must be considered in calculating radiation interchange. This procedure is not currently used because of heat transfer specialists' greater familiarity with opaque materials, and because the required optical data are not generally available.

Two important optical parameters are defined by the Hamaker relations<sup>9</sup>: the absorption coefficient ( $a$ ), and the scattering coefficient ( $s$ ). The former is basically an intrinsic property that is related to, but not equal to ( $\alpha$ ), the absorption coefficient of single crystals. The absorption coefficient ( $a$ ) is partially dependent on the structurally sensitive scattering coefficient.

In order to be able to calculate radiant energy emissivity from non-isothermal semi-transparent materials, values of ( $a$ ) and ( $s$ ) are needed as a function of temperature and wavelength. The reflectivity and isothermal emissivity of infinitely thick samples allow the determination of the ratio ( $a/s$ ) but not the individual values. Thus, present emissivity values cannot be converted into the required coefficients. Measurements of the reflectivity, transmissivity, or absorptivity of a given material as a function of thickness does allow both constants to be determined. In general, it is most convenient and accurate to measure transmissivity for scattering systems; although for very thin layers, completely diffuse radiation is required, and for thick layers the accuracy decreases due to loss of signal strength of the transmitted beam. Because radiation emissivity differences between isothermal and non-isothermal specimens increase with increasing temperature, data for optical coefficients is most difficult to obtain in the region of greatest interest. Since ( $s$ ) is nearly temperature independent, this problem can best be overcome by elucidating the interrelationships between ( $a$ ), ( $s$ ) and ( $\alpha$ ) which are more easily measured at or extrapolated to high temperature.

The differential equations presented by Hamaker provide a basis for developing analytical expressions of broad applicability. In contrast to some other analytical systems, these derived expressions are general enough to include the strong effect of scattering in addition to absorption effect. For real systems, the relative simplicity of the equations facilitates a study of mechanisms and trends and a determination of design data. Although the derived equations are not entirely general, a number of important cases may be considered and certain optical constants may be calculated for a particular material. Other analytical systems, particularly numerical methods, may provide greater precision since some general assumptions were made for deviation.

We have modified Hamaker's original differential equations by introducing a correction for the radiant energy density. This correction appears as  $(n^2)$ , the index of refraction squared, in both the differential equations and the general solutions. An additional correction for surface reflection is included in the boundary conditions.

Further development of the Hamaker relations to a form suitable for heat transfer calculations depends on a clear understanding of the underlying assumptions and the conditions under which they can be used. Two important assumptions involve diffuse radiation impinging on the body and the nature of scattering produced by a pore phase. In addition, the equations contain an approximation for the conductive heat transfer which may impair accuracy under some conditions. Methods have been considered and developed for evaluation of limits arising from approximations and assumptions inherent to this system of equations. For some of the simpler cases, new analytical relations have been derived.

### 3.1 Further Development of the Hamaker Relations

The method used is based on a system originally conceived by Schuster<sup>(10,11)</sup> and added to by Hamaker<sup>(9)</sup>; the notation used here is essentially that of Hamaker. The basic method is that of dividing the flux into two parts: one flowing in a positive direction, and the other in a negative direction. A set of simultaneous differential equations is used to describe these fluxes and the other necessary parameters. Since only a forward and a backward flux are considered this is a one-dimensional calculation and therefore has as a basic assumption that the incident radiation is diffuse (i.e., the intensity is equal for all angles of incidence), and that the radiation scattered sideways is compensated for by an equal contribution from neighbouring parts of the layer (i.e., the area investigated is either small in cross section compared with the total illuminated cross section of the sample or is large compared to the thickness of the sample). This condition is not a severe limitation since most practical heat transfer problems are concerned with diffuse radiation.

The treatment for the situations where temperature gradients are present suffers from the further limitation that only total radiation is considered and therefore the fact that the wavelength distribution of black-body emission changes with temperature is not taken into account. Also it is assumed that the properties of the material change only gradually. This then implies the assumption that the temperature gradient across the sample which is being measured is small. Practically all the methods of calculation in use today also suffer from this limitation and in practice there are calculation schemes which can be used to alleviate the problem.

#### 3.1.1 Isothermal Layers, General Solutions

The total radiant flux is divided into two parts:

$I$  = The flux in the direction of the positive  $x$  axis

$J$  = The flux in the direction of the negative  $x$  axis

An absorption coefficient,  $(a)$ , is defined by requiring that  $(a I dx)$  be the amount of the radiation absorbed from the flux  $(I)$  on passing through an infinitesimal layer,  $(dx)$ ; a scattering coefficient  $(s)$ , is similarly defined by requiring that the flux scattered backward from  $(I)$  (and therefore added to  $J$ ) in an infinitesimal layer  $(dx)$  is  $(s I dx)$ . On passing through this layer  $(I)$  will then be diminished by the amount absorbed and the amount scattered, but will be increased by the flux lost by scattering from  $(J)$ , or:

$$dI/dx = - (a+s) I + sJ \quad (1)$$

similarly:

$$dJ/dx = (a+s) J - sI \quad (2)$$

The general solutions of these equations can be found by putting

$$I = C_1 e^{\sigma x} + C_2 e^{-\sigma x} \quad (3)$$

$$J = C_3 e^{\sigma x} + C_4 e^{-\sigma x} \quad (4)$$

only two of the four constants  $C_1, \dots, C_4$  being arbitrary. The solutions (using the same notation as Hamaker) are then:

$$I = A (1-\beta_0) e^{\sigma_0 x} + B (1+\beta_0) e^{-\sigma_0 x} \quad (5)$$

$$J = A (1+\beta_0) e^{\sigma_0 x} + B (1-\beta_0) e^{-\sigma_0 x} \quad (6)$$

where

$$\sigma_0 = \sqrt{a(a+2s)} \quad (7)$$

$$\beta_0 = \sqrt{a/(a+2s)} = \sigma_0/(a+2s) \quad (8)$$

both roots being taken with a positive sign. In these equations (A) and (B) are constants to be determined by the boundary conditions.

### 3.1.2 Isothermal Layers; Specific Solutions With No Diffusing Screen

One of the cases for which one would like specific solutions is that of a layer placed in a beam of diffuse radiation where there is reflection from both internal and external surfaces.

At an interface where the index of refraction is increasing, let the reflectivity equal ( $\rho_0$ ). At an interface where the index of refraction is decreasing, let the reflectivity equal ( $\rho_i$ ). (The actual values of these constants can be calculated from the index of refraction and will be discussed in a following section.)

The following nomenclature will be used (where D is the thickness of a layer):

$I_i$  = The incident flux at  $x = 0$

$I_0$  = The forward flux immediately below the interface  
 $x = 0$

$J_0$  = The backward flux immediately below the interface  
 $x = 0$

$J_D$  = The backward flux immediately below the inter-  
face  $x = D$

$I_D$  = The forward flux immediately below the inter-  
face  $x = D$

There is assumed to be no incident flux on the back surface  $x = D$ .

Then the boundary conditions are that at the front surface ( $x = 0$ ), part ( $\rho_o$ ) of the incident radiation ( $I_i$ ) is reflected back, and part ( $1-\rho_o$ ) is transmitted. The flux immediately below this interface ( $I_o$ ) is composed of this flux ( $[1-\rho_o]I_i$ ) plus that flux reflected from the inner surface of  $x = 0$  or ( $\rho_i J_o$ ).

or (at  $x = 0$ ):

$$I_o = (1-\rho_o)I_i + \rho_i J_o \quad (9)$$

At the back surface ( $x = D$ ) since there is no incident radiation, the only flux is that part ( $\rho_i$ ) reflected from the remaining forward flux ( $I_D$ ).

or (at  $x = D$ ):

$$J_D = \rho_i I_D \quad (10)$$

Substituting in these equations for  $I_o$ ,  $J_o$ ,  $I_D$ , and  $J_D$  from equations (5) and (6), gives:

$$A(1-\beta_o) + B(1+\beta_o) = (1-\rho_o)I_i + \rho_i A(1+\beta_o) + \rho_i B(1-\beta_o) \quad (11)$$

and

$$A(1+\beta_o)e^{\sigma_o D} + B(1-\beta_o)e^{-\sigma_o D} = \rho_i A(1-\beta_o)e^{\sigma_o D} + \rho_i B(1+\beta_o)e^{-\sigma_o D} \quad (12)$$

These are the equations to be solved for the constants (A) and (B) for these particular boundary conditions. They are (when the exponentials are substituted for by hyperbolic functions):

$$A = \frac{I_i e^{-\sigma_o D} (1-\rho_o) [\beta_o (1+\rho_i) - (1-\rho_i)]}{2 \{ [\beta_o^2 (1+\rho_i)^2 + (1-\rho_i)^2] \sinh \sigma_o D + 2 \beta_o (1-\rho_i)^2 \cosh \sigma_o D \}} \quad (13)$$

$$B = \frac{I_i e^{\sigma_o D} (1-\rho_o) [\beta_o (1+\rho_i) + (1-\rho_i)]}{2 \{ [\beta_o^2 (1+\rho_i)^2 + (1-\rho_i)^2] \sinh \sigma_o D + 2 \beta_o (1-\rho_i)^2 \cosh \sigma_o D \}} \quad (14)$$

using these values in equations (5) and (6) gives the following expressions for  $I_x$  (the forward flux at  $x$ ) and  $J_x$  (the backward flux at  $x$ ):

$$I_x = \frac{I_i (1-\rho_o) \{ [\beta_o (1+\rho_i) + (1-\rho_i)] (1+\beta_o) e^{-\sigma_o x} e^{\sigma_o D} + [\beta_o (1+\rho_i) - (1-\rho_i)] (1-\beta_o) e^{\sigma_o x} e^{-\sigma_o D} \}}{2 \{ [\beta_o^2 (1+\rho_i)^2 + (1-\rho_i)^2] \sinh \sigma_o D + 2 \beta_o (1-\rho_i)^2 \cosh \sigma_o D \}} \quad (15)$$

$$J_x = \frac{I_i (1-\rho_o) \{ [\beta_o (1+\rho_i) + (1-\rho_i)] (1-\beta_o) e^{-\sigma_o x} e^{\sigma_o D} + [\beta_o (1+\rho_i) - (1-\rho_i)] (1+\beta_o) e^{\sigma_o x} e^{-\sigma_o D} \}}{2 \{ [\beta_o^2 (1+\rho_i)^2 + (1-\rho_i)^2] \sinh \sigma_o D + 2 \beta_o (1-\rho_i)^2 \cosh \sigma_o D \}} \quad (16)$$

In practice, it is impossible to check these quantities experimentally; what can be checked, however, is the trans-

missivity and the reflectivity. To arrive at these qualities we first determine the forward flux immediately under the back surface ( $I_D$ ) by substituting (D) for (x) in equation (15).

Then:

$$I_D = \frac{I_i 2\beta_o (1-\rho_o)}{[\beta_o^2 (1+\rho_i)^2 + (1-\rho_i)^2] \sinh \sigma_o D + 2\beta_o (1-\rho_i)^2 \cosh \sigma_o D} \quad (17)$$

The transmissivity ( $\tau$ ) is then the ratio of the amount of radiation of the above that gets through the surface ( $[1-\rho_i]I_D$ ) to the incident radiation.

Or:

$$\tau = I_D (1-\rho_i) / I_i \quad (18)$$

giving for the transmissivity:

$$\tau = \frac{2\beta_o (1-\rho_o) (1-\rho_i)}{[\beta_o^2 (1+\rho_i)^2 + (1-\rho_i)^2] \sinh \sigma_o D + 2\beta_o (1-\rho_i)^2 \cosh \sigma_o D} \quad (19)$$

The reflectivity ( $r$ ) can be found similarly by adding to the fraction of incident radiation reflected from the front surface ( $\rho_o I_i$ ) the amount of backward flux that gets through the interface ( $[1-\rho_i]J_o$ ).

Then:

$$r = \frac{[(1-\rho_i)^2 - \beta_o^2 (1-\rho_i - 2\rho_o) (1+\rho_i)] \sinh \sigma_o D + 2\beta_o (\rho_o + \rho_i) (1-\rho_i) \cosh \sigma_o D}{[\beta_o^2 (1+\rho_i)^2 + (1-\rho_i)^2] \sinh \sigma_o D + 2\beta_o (1-\rho_i)^2 \cosh \sigma_o D} \quad (20)$$

It is also possible to calculate the absorptivity ( $\alpha$ ) of the layer since  $\alpha + \rho + \tau = 1$ . It is:

$$\alpha = \frac{2\beta_o (1-\rho_o) [\beta_o (1+\rho_i) \sinh \sigma_o D + (1-\rho_i) \{\cosh \sigma_o D - 1\}]}{[\beta_o^2 (1+\rho_i)^2 + (1-\rho_i)^2] \sinh \sigma_o D + 2\beta_o (1-\rho_i)^2 \cosh \sigma_o D} \quad (21)$$

This is also the emission of the layer relative to black body radiation according to Kirchhoff's law.

### 3.1.3 Determining Optical Constants From Transmissivity Measurements

One of the objects of making transmissivity measurements is to use them to calculate optical constants of the material. In order to do this, equation (19) for the transmissivity of the material has to be solved for the constants. Cross multiplying in the equation (19) gives:

$$2\beta_o (1-\rho_o) (1-\rho_i) = \tau \beta_o^2 (1+\rho_i)^2 \sinh \sigma_o D + \tau (1-\rho_i)^2 \sinh \sigma_o D + 2\beta_o \tau (1-\rho_i)^2 \cosh \sigma_o D \quad (22)$$

Or, by regrouping the terms:

$$\tau (1-\rho_i)^2 \sinh \sigma_o D = \beta_o \{2[(1-\rho_o) (1-\rho_i) - \tau (1-\rho_i)^2 \cosh \sigma_o D] - \tau \beta_o (1+\rho_i)^2 \sinh \sigma_o D\} \quad (23)$$

If we consider two layers of thicknesses ( $D_1$ ) and ( $D_2$ ) and transmissivities ( $\tau_1$ ) and ( $\tau_2$ ) respectively then (dividing by equal quantities):

$$\frac{\tau_1 (1-\rho_i)^2 \sinh \sigma_0 D_1}{\tau_2 (1-\rho_i)^2 \sinh \sigma_0 D_2} = \quad (24)$$

$$\frac{2[(1-\rho_0)(1-\rho_i) - \tau_1 (1-\rho_i)^2] \cosh \sigma_0 D_1 - \tau_1 \beta_0 (1+\rho_i)^2 \sinh \sigma_0 D_1}{2[(1-\rho_0)(1-\rho_i) - \tau_2 (1-\rho_i)^2] \cosh \sigma_0 D_2 - \tau_2 \beta_0 (1+\rho_i)^2 \sinh \sigma_0 D_2}$$

or (again cross multiplying):

$$\begin{aligned} & 2 \tau_1 (1-\rho_0)(1-\rho_i)^3 \sinh \sigma_0 D_1 - 2 \tau_1 \tau_2 (1+\rho_i)(1-\rho_i)^3 \sinh \sigma_0 D_1 \cosh \sigma_0 D_2 \\ & \quad - \beta_0 \tau_1 \tau_2 (1+\rho_i)^2 (1-\rho_i)^2 \sinh \sigma_0 D_1 \sinh \sigma_0 D_2 = \\ & 2 \tau_2 (1-\rho_0)(1-\rho_i)^3 \sinh \sigma_0 D_2 - 2 \tau_1 \tau_2 (1+\rho_i)(1-\rho_i)^3 \sinh \sigma_0 D_2 \cosh \sigma_0 D_1 \\ & \quad - \beta_0 \tau_1 \tau_2 (1+\rho_i)^2 (1-\rho_i)^2 \sinh \sigma_0 D_1 \sinh \sigma_0 D_2 \end{aligned} \quad (25)$$

All the terms involving ( $\beta_0$ ) drop out of the above equation giving (having made use of the identity  $\sinh [x - y] \equiv \sinh x \cosh y - \cosh x \sinh y$ ):

$$\frac{\sinh \sigma_0 D_1}{\tau_2} - \frac{\sinh \sigma_0 D_2}{\tau_1} = \frac{(1+\rho_i)}{(1-\rho_0)} \sinh \sigma_0 (D_1 - D_2) \quad (26)$$

Or, if sample thicknesses are chosen such that

$$D_1 = 2 D_2 = 2D \quad (27)$$

Then:

$$\frac{\sinh 2 \sigma_o D}{\tau_2} - \frac{\sinh \sigma_o D}{\tau_1} = \frac{(1+\rho_i)}{(1-\rho_o)} \sinh \sigma_o D \quad (28)$$

$$\text{but } \sinh 2x = 2 \sinh x \cosh x \quad (29)$$

and:

$$\frac{2 \sinh \sigma_o D \cosh \sigma_o D}{\tau_2} - \frac{\sinh \sigma_o D}{\tau_1} = \frac{(1+\rho_i)}{(1-\rho_o)} \sinh \sigma_o D \quad (30)$$

$$\text{and } \cosh \sigma_o D = \frac{\tau_2 [\tau_1 (1+\rho_i) + (1-\rho_o)]}{2 \tau_1 (1-\rho_o)} \quad (31)$$

allowing one to calculate  $(\sigma_o)$  from two transmissivity measurements.

Once  $(\sigma_o)$  is known,  $(\beta_o)$  can be found either by solving equation (19) by the quadratic formula, or with an electronic data processing machine.

#### 3.1.4 Isothermal Layers With Diffusing Screen Corrections.

In order to approximate the theoretical conditions in the experimental measurements, a screen can be used before the sample to diffuse the radiation. Since this screen will reflect back onto the sample some of the radiation which the

sample reflects, a correction must be made for it. The flux incident on the sample is, however, the initial flux passing through the screen ( $I_h$ ) enhanced by the radiation rereflected from the screen. If  $I_i$  is the total flux in this case and ( $\rho_s$ ) is the reflectivity of the screen, (and the reflectivity of the sample ( $\rho$ ) is the same as that calculated in the previous section),

Then:

$$I_i = I_h + \rho \rho_s I_h + \rho^2 \rho_s^2 I_h + \rho^3 \rho_s^3 I_h \dots \quad (32)$$

or

$$I_i = I_h \sum_{n=0}^{\infty} (\rho \rho_s)^n \quad (33)$$

but

$$\sum_{n=0}^{\infty} x^n = 1/(1-x) \text{ for } 0 < x < 1 \quad (34)$$

$$\text{since both } 0 < \rho < 1 \text{ and } 0 < \rho_s < 1, 0 < \rho \rho_s < 1 \quad (35)$$

$$\text{and } I_i = I_h / (1 - \rho \rho_s)$$

If we introduce this result into equation (19) for the transmissivity we obtain the following result where ( $\tau$ ) in this case is now the ratio of the transmitted flux to the flux measured from the screen with the sample removed.

$$\tau = \frac{2 \beta_o (1 - \rho_o) (1 - \rho_i)}{(1 - \rho \rho_s) \{ [\beta_o^2 (1 + \rho_i)^2 + (1 - \rho_i)^2] \sinh \sigma_o D + 2 \beta_o (1 - \rho_i)^2 \cosh \sigma_o D \}} \quad (36)$$

If the value of ( $\rho$ ) from equation (20) is introduced into equation (36) one obtains:

$$\tau = \frac{2\beta_o (1-\rho_o)(1-\rho_i)}{\{\beta_o^2(1+\rho_i) [1+\rho_i+\rho_s(1-\rho_i-2\rho_o)] + (1-\rho_i)^2(1-\rho_s)\} \sinh \sigma_o D + 2\beta_o(1-\rho_i) [1+\rho_i-\rho_s(\rho_o+\rho_i)] \cosh \sigma_o D} \quad (37)$$

This equation can then be solved for  $(\sigma_o)$  [by steps similar to equations (22) - (31)]:

$$\frac{\sinh \sigma_o D_1}{\tau_2} - \frac{\sinh \sigma_o D_2}{\tau_1} = \frac{[1+\rho_i-\rho_s(\rho_o+\rho_i)]}{(1-\rho_o)} \sinh \sigma_o (D_1 - D_2) \quad (38)$$

where  $(\tau_1)$  and  $(\tau_2)$  are the transmissivities (as measured by the methods described here) of two layers of thicknesses  $(D_1)$  and  $(D_2)$  respectively.

Or if thicknesses are chosen such that:

$$D_1 = 2D_2 \equiv 2D \quad (39)$$

$$\cosh \sigma_o D = \frac{\tau_2 \{ \tau_1 [1+\rho_i-\rho_s(\rho_o+\rho_i)] + (1-\rho_o) \}}{2 \tau_1 (1-\rho_o)} \quad (40)$$

### 3.1.5 Determination of the Value of $\sigma_o$ and $\beta_o$ . From Experimentally Obtained Values of Transmissivity

From equation (38),  $\sigma_o$  can be determined from measurement of the total reflectivity of a surface where the index of refraction is increasing,  $\rho_o$ , and the total reflectivity when the index of refraction is decreasing,  $\rho_i$ , the reflectivity of a diffusing screen placed before the sample,  $\rho_s$ , and the transmissivity  $\tau_1$  and  $\tau_2$ , of layers of thicknesses,  $D_1$  and  $D_2$  respectively.

Since the solution of this equation by hand is laborious and time consuming, a program has been obtained to solve it by machine. The Fortran statements of this program are given in Appendix I. This program was written to be run on an IBM 7090 electronic data processing system. The other material constant, the corrected emissivity for transfer between two parallel plates,  $\beta_0$ , can then be obtained from the expression for  $\tau$  given in equation (37).

A Fortran program to solve this equation for  $\beta_0$  using the value of  $\sigma_0$  obtained above and a value of  $\tau$  and  $D$  measured, and the known values of the other constants is given in Appendix II. These two programs allow one to obtain the materials constants  $\beta_0$  and  $\sigma_0$  from transmissivity measurements of two or more layers of different thicknesses. These programs have been debugged and tested and the answers they produce agree with answers previously obtained by hand.

### 3.1.6 Surface Reflections

In order to use the result derived here for practical calculations, it is necessary to know the value of the reflectivity at the interfaces. The case of interest here is that of diffuse radiation. Then the intensity is equal for any angle of incidence, and the flux is then the intensity times the cosine of the angle of incidence (as measured from the angle to the normal).

If  $\theta$  is the angle of incidence, the portion of radiation coming from that part of the hemisphere between  $\theta$  and  $\theta + d\theta$  is  $\sin \theta d\theta$  (times a constant factor which will be normalized). The total flux at this angle of incidence is then  $\cos \theta \sin \theta d\theta$ . If  $\rho_\theta$  is the reflectivity of the interface for a ray at angle  $\theta$ , then  $\rho_\theta \cos \theta \sin \theta d\theta$  is the amount reflected for each incremental angle. To find the total reflectivity then, one would integrate this expression and divide by the total radiation ( $\int_0^{\pi/2} \sin \theta \cos \theta d\theta$ ) (since we are considering here only an interface where the index of refraction is increasing, this reflectivity is  $\rho_0$ ):

$$\rho_o = \frac{\int_0^{\pi/2} \rho_\theta \sin \theta \cos \theta d\theta}{\int_0^{\pi/2} \sin \theta \cos \theta d\theta} \quad (41)$$

The value of  $(\rho_\theta)$  is given by the following Fresnel equation for reflection of randomly polarized light where  $(\theta)$  is the angle of incidence,  $(\gamma)$  is the angle of refraction,  $(\sin \theta = n \sin \gamma)$ , and  $(n)$  is the index of refraction.

$$\rho_\theta = \frac{1}{2} \left[ \frac{\sin^2(\theta - \gamma)}{\sin^2(\theta + \gamma)} + \frac{\tan^2(\theta - \gamma)}{\tan^2(\theta + \gamma)} \right] \quad (42)$$

Therefore:

$$\rho_o = \frac{\int_0^{\pi/2} \sin \theta \cos \theta [\sin^2(\theta - \gamma) / \sin^2(\theta + \gamma) + \tan^2(\theta - \gamma) / \tan^2(\theta + \gamma)] d\theta}{2 \int_0^{\pi/2} \sin \theta \cos \theta d\theta} \quad (43)$$

This integration has been carried out by Walsh<sup>(12)</sup> with the result that:

$$\begin{aligned} \rho_o = & \frac{1}{2} + \frac{(n-1)(3n+1)}{6(n+1)^2} + \frac{[n^2(n^2-1)^2]}{[(n^2+1)^3]} \log \frac{(n-1)}{(n+1)} \\ & - \frac{2n^3(n^2+2n-1)}{(n^2+1)(n^4-1)} + \frac{[8n^4(n^4+1)]}{[(n^2+1)(n^4-1)^2]} \log n \end{aligned} \quad (44)$$

which allows one to calculate the reflectivity of an interface for diffuse radiation as a function of the index of refraction. These calculations have been carried out by Ryde and Cooper<sup>(13)</sup> with the results shown in Table I; this table compares the total reflectivity for diffuse radiation to that at perpendicular incidence.

At an interface where the index of refraction is decreasing, the situation is more complex. Here any radiation arriving at an angle to the normal greater than  $\sin^{-1}(1/n)$  is totally reflected. To find this fraction we would integrate from this angle to  $(\pi/2)$  and divide by the total flux:

$$\text{total reflectivity} = \frac{\int_{\sin^{-1}(1/n)}^{\pi/2} \sin \theta \cos \theta \, d\theta}{\int_0^{\pi/2} \sin \theta \cos \theta \, d\theta} \quad (45)$$

since

$$\int \sin \theta \cos \theta \, d\theta = \frac{1}{2} \sin^2 \theta \quad (46)$$

$$\begin{aligned} \text{Total reflectivity} &= 1 - \frac{1}{n^2} \\ &= \frac{n^2 - 1}{n^2} \end{aligned} \quad (47)$$

Of the flux that is transmitted,  $(1 - \text{reflectivity})$  or  $(1/n^2)$ , a portion of this is reflected by the Fresnel reflection discussed above. This can be shown to be the same fraction as that previously calculated ( $\rho_0$ ). The sum of these two is then the total reflectivity at a surface where the index of refraction is decreasing ( $\rho_i$ ):

TABLE 1

Comparison of the total reflectivity for  
diffuse radiation to that at perpendicular incidence  
(After Ryde and Cooper (13))

<u>Index of Refraction</u>	<u>Reflectivity for Perpendicular Incidence</u>	<u>Reflectivity for Diffuse Radiation</u>
1.00	0	0
1.1	0.0023	0.026
1.15	0.0049	0.035
1.2	0.0083	0.045
1.25	0.012	0.053
1.3	0.017	0.061
1.35	0.022	0.069
1.4	0.028	0.077
1.45	0.033	0.085
1.5	0.040	0.092
1.55	0.047	0.100
1.6	0.053	0.107
1.65	0.060	0.114
1.7	0.067	0.121
1.8	0.082	0.134
1.9	0.096	0.146

$$\rho_i = \frac{n^2 - 1}{n^2} + \frac{\rho_o}{n^2} \quad (48)$$

Or:

$$\rho_i = \frac{n^2 - 1 + \rho_o}{n^2} \quad (49)$$

This can be an important (though usually neglected) factor in heat transfer calculations. For instance, for a material with an index of refraction of 1.5,  $\rho_i$  would be 0.595. For a material of index of refraction 2.0,  $\rho_i$  would be 0.788, both factors being quite significant.

Rearranging equation (49),

$$1 - \rho_i = \frac{1 - \rho_o}{n^2} \quad (50)$$

showing that the transmission through an interface where the index of refraction is decreasing is only one divided by  $n^2$  of the transmission of an interface where the index of refraction is increasing.

### 3.1.7 Non-Isothermal Layers

In order to be useful in heat transfer calculations this theory must be extended to non-isothermal situations. This can be done (as is shown by Hamaker and also by Schuster) by adding a term in each radiation equation representing the amount of energy emitted by the infinitesimal region. This is  $(\epsilon E dx)$  where  $(\epsilon)$  is the emissivity and  $(E)$  is the black-body radiation at the temperature at  $(x)$ . Making use of

Kirchhoff's law, this term becomes  $(a E dx)$  where  $(a)$  is the previously defined absorption coefficient. An additional heat balance equation is now needed expressing the fact that heat is neither accumulated nor produced within the body:

$$\frac{k d^2 T}{dx^2} + a(I+J) = 2 a E \quad (51)$$

where  $(k)$  is the lattice thermal conductivity. The first term on the left side represents the heat accumulated by conduction; the second term is the heat absorbed from the radiation, and the sum of these equals the heat loss by radiation (the term on the right).

The total black-body radiation within a media of refractive index  $(n)$  is given by the Stefan-Boltzmann equation:

$$E = \sigma' n^2 T^4 \quad (52)$$

where  $(\sigma')$  is the Stefan-Boltzmann radiation constant and  $(T)$  is the absolute temperature. If the temperature is high and the temperature gradient not too large then  $(E)$  may be represented by:

$$E = E_0 + b (T - T_0) \quad (53)$$

where

$$b = 4 \sigma' n^2 T_0^3 \quad (54)$$

and  $(T_0)$  is a temperature close to the actual temperature, and  $(E_0)$  is the corresponding total radiation. When the above equation holds the temperature may be fixed equally as well by  $(E)$  as by  $(T)$  and, since this simplifies matters,  $(E)$  has been retained in the equations rather than  $(T)$ . The

set of simultaneous differential equations is then:

$$\frac{dI}{dx} = -(a+s) I + sJ + aE \quad (55)$$

$$\frac{dJ}{dx} = (a+s) J - sI - aE \quad (56)$$

$$\frac{k}{b} \cdot \frac{d^2 E}{dx^2} + a(I+J) = 2aE \quad (57)$$

Hamaker shows that the complete general solution of these equations is:

$$I = A(1-\beta)e^{\sigma x} + B(1+\beta)e^{-\sigma x} + C(\sigma x - \beta) + F \quad (58)$$

$$J = A(1+\beta)e^{\sigma x} + B(1-\beta)e^{-\sigma x} + C(\sigma x + \beta) + F \quad (59)$$

$$E = -A\kappa e^{\sigma x} - B\kappa e^{-\sigma x} + C\sigma x + F \quad (60)$$

$$\text{where } \sigma = +\sqrt{\frac{2ab}{k} + a(a+2s)} = \sigma_0 \sqrt{(1+\kappa)} \quad (61)$$

$$\beta = \sigma/(a+2s) \quad (62)$$

$$\kappa = 2b/k(a+2s) = 2b\beta/k_0 \quad (63)$$

And we have introduced here the proper  $(n^2)$  term which does not appear in Hamaker's work.

To illustrate how this theory might be used, we will derive the particular solutions for a layer receiving radiation at both surfaces, and where heat is being conducted away

from the surfaces. The amount of heat being conducted away from the surface must equal that conducted to the surface in the solid giving one boundary condition at each surface. The other two boundary conditions are supplied by the radiation interchange at the surface. The temperature (particularly at the surfaces) and the emitted fluxes will be solved for.

Using the same notation as the previous sections, immediately below the front surface  $x=0$ , the forward flux ( $I_o$ ) is equal to that part of the incident flux ( $I_i$ ) which isn't reflected ( $[1-\rho_o]I_i$ ) plus the amount of the backward flux at this surface ( $J_o$ ) which was reflected ( $\rho_i J_o$ ).

Therefore;

$$I_o = (1-\rho_o) I_i + \rho_i J_o \quad (64)$$

Or substituting from equations (58) and (59):

$$\begin{aligned} A(1-\beta) + B(1+\beta) - C\beta + F &= (1-\rho_o)I_i + \rho_i A(1+\beta) + \rho_i B(1-\beta) \\ &\quad + \rho_i C\beta + F\rho_i \end{aligned} \quad (65)$$

Similarly, immediately below the back surface  $x=D$ , the backward flux ( $J_D$ ) is composed of the part of the incident flux on this surface ( $J_i$ ) which is transmitted ( $[1-\rho_o]J_i$ ) plus the part of the forward flux at this surface which is reflected ( $\rho_i I_D$ ):

And:

$$J_D = (1-\rho_o)J_i + \rho_i I_D \quad (66)$$

or:

$$A(1+\beta)e^{\sigma D} + B(1-\beta)e^{-\sigma D} + C(\sigma D + \beta) + F =$$

$$(1-\rho_o)J_i + \rho_i A(1-\beta)e^{\sigma D} + B\rho_i(1+\beta)e^{-\sigma D} + \rho_i C(\sigma D - \beta) + \rho_i F$$
(67)

If we define ( $\eta$ ) as being equal to the gradient at the surface times ( $b$ ):

$$\eta \equiv b \left( \frac{dT}{dx} \right)_{\text{surface}} = \left( \frac{dE}{dx} \right)_{\text{surface}} \quad (68)$$

Or, if the heat is conducted away by a gas:

$$\eta \equiv \left( \frac{dE}{dx} \right)_{\text{surface}} = \frac{-b Q_g}{k} \quad (69)$$

where ( $Q_g$ ) is the heat being conducted (or convected) away by the gas.

Then since (by differentiating equation (60)):

$$\frac{dE}{dx} = -A \kappa \sigma e^{\sigma x} + B \kappa \sigma e^{-\sigma x} + C \sigma \quad (70)$$

The other two boundary conditions are:

$$\eta = \left( \frac{dE}{dx} \right)_o \quad (71)$$

$$\eta = -A \kappa \sigma + B \kappa \sigma + C \sigma \quad (72)$$

and

$$\eta = \left( \frac{dE}{dx} \right)_D \quad (73)$$

$$\eta = -A \kappa \sigma e^{\sigma D} + B \kappa \sigma e^{-\sigma D} + C \sigma \quad (74)$$

The four simultaneous equations (65), (67), (72) and (74) are then solved for the constants A, B, C, and F where it has been found convenient to define a function consisting of the denominator:

let:

$$2(1-\rho_i)(\cosh \sigma D - 1) + [2\beta(1+\rho_i)(1+\kappa) + \kappa \sigma D(1-\rho_i)] \sinh \sigma D \equiv \text{etc} \quad (75)$$

then:

$$A = \frac{(e^{-\sigma D} - 1) \{ \sigma(1-\rho_o)(I_i - J_i) + \eta[2\beta(1+\rho_i) + \sigma D(1-\rho_i)] \}}{2 \sigma \text{ etc}} \quad (76)$$

$$B = \frac{(e^{\sigma D} - 1) \{ \sigma(1-\rho_o)(I_i - J_i) + \eta[2\beta(1+\rho_i) + \sigma D(1-\rho_i)] \}}{2 \sigma \text{ etc}} \quad (77)$$

$$C = \frac{-2 \kappa \sigma(1-\rho_o)(I_i - J_i) \sinh \sigma D + 4\eta[(1-\rho_i)(\cosh \sigma D - 1) + \beta(1+\rho_i) \sinh \sigma D]}{2 \sigma \text{ etc}} \quad (78)$$

$$F = \frac{2\sigma n^2 I_i \{ (1-\rho_i)(\cosh \sigma D - 1) + [\beta(1+\rho_i)(1+\kappa) + \kappa \sigma D(1-\rho_i)] \sinh \sigma D \} + 2\sigma n^2 J_i \{ (1-\rho_i)(\cosh \sigma D - 1) + \beta(1+\rho_i)(1+\kappa) \sinh \sigma D \} - 2\sigma \eta D \{ (1-\rho_i)(\cosh \sigma D - 1) + \beta(1+\rho_i) \sinh \sigma D \}}{2 \sigma \text{ etc}} \quad (79)$$

By introducing these constants into equations (58), (59), and (60) it is now possible to find the fluxes and temperature at any point in terms of the radiant and thermal fluxes.

They are:

$$I = \frac{\sigma(1-\rho_0)(I_i - J_i) \{e^{\sigma x}(e^{-\sigma D} - 1)(1-\beta) + (e^{\sigma D} - 1)(1+\beta)e^{-\sigma x} - 2\kappa(\sigma x - \beta) \sinh \sigma D\} + \eta \{ [2\beta(1+\rho_i) + \sigma D(1-\rho_i)] [e^{\sigma x}(e^{-\sigma D} - 1)(1-\beta) + e^{-\sigma x}(e^{\sigma D} - 1)(1+\beta)] + 4(\sigma x - \beta) [\beta(1+\rho_i) \sinh \sigma D + (1-\rho_i)(\cosh \sigma D - 1)] \}}{2 \sigma \text{ etc} + F} \quad (80)$$

$$J = \frac{\sigma(1-\rho_0)(I_i - J_i) \{e^{\sigma x}(e^{-\sigma D} - 1)(1+\beta) + e^{-\sigma x}(e^{\sigma D} - 1)(1-\beta) - 2\kappa(\sigma x + \beta) \sinh \sigma D\} + \eta \{ [2\beta(1+\rho_i) + \sigma D(1-\rho_i)] [e^{\sigma x}(e^{-\sigma D} - 1)(1+\beta) + e^{-\sigma x}(e^{\sigma D} - 1)(1-\beta)] + 4(\sigma x + \beta) [\beta(1+\rho_i) \sinh \sigma D + (1-\rho_i)(\cosh \sigma D - 1)] \}}{2 \sigma \text{ etc} + F} \quad (81)$$

$$E = \frac{-\kappa \sigma(1-\rho_0)(I_i - J_i) \{e^{\sigma x}(e^{-\sigma D} - 1) + e^{-\sigma x}(e^{\sigma D} - 1) + 2\sigma x \sinh \sigma D\} - \eta \{ \kappa [2\beta(1+\rho_i) + \sigma D(1-\rho_i)] [e^{\sigma x}(e^{-\sigma D} - 1) + e^{-\sigma x}(e^{\sigma D} - 1)] - 4\sigma x [\beta(1+\rho_i) \sinh \sigma D + (1-\rho_i)(\cosh \sigma D - 1)] \}}{2 \sigma \text{ etc} + F} \quad (82)$$

These are again not measurable quantities. The quantities desired are the fluxes emitted at each surface and the temperatures at the surfaces. At the back of the layer, ( $x=D$ ), the flux emitted in the forward direction (here denoted by  $I_e$ ) is equal to the fraction of the forward flux immediately under this surface ( $I_D$ ) which is not reflected at this surface ( $[1-\rho_i]I_D$ ) plus the fraction of the incident radiation on this surface which is reflected into the forward direction ( $\rho_o J_i$ ).

or

$$I_e = (1-\rho_i)I_D + \rho_o J_i \quad (83)$$

Similarly (where  $J_e$  is the flux emitted in the backward direction at the front surface):

$$J_e = (1-\rho_i) J_o + \rho_o I_i \quad (84)$$

Finally the energy equivalent of the temperature at the surfaces is found by substituting  $x=0$  and  $x=D$  into equation (82). Actually in the constant (b), the ( $n^2$ ) term should be the index of refraction of the material in which the particular quantity is measured. In order to keep the notation consistent in this part, the ( $n^2$ ) term will be kept in the constant (b) but the energy equivalent of temperature measured outside the sample (here denoted by  $E_{ao}$  and  $E_{aD}$ ) will be divided by ( $n^2$ ) so that the numerical results will be correct. Before setting down the results it is desirable to define the following functions since most of the equations are symmetrical.

$$\text{Let etc.} \equiv 2(1-\rho_i)(\cosh \sigma D - 1) + [2\beta(1+\rho_i)(1+\kappa) + \kappa\sigma D(1-\rho_i)] \sinh \sigma D \quad (85)$$

$$f_1 \equiv \frac{2\beta(1-\rho_o)(1+\kappa) \sinh \sigma D}{\text{etc}} \quad (86)$$

$$f_2 \equiv \frac{2(1-\rho_i)(\cosh \sigma D - 1) + [2\beta(\rho_o + \rho_i)(1+\kappa) + \kappa \sigma D(1-\rho_i)] \sinh \sigma D}{\text{etc}} \quad (87)$$

$$f_3 \equiv \frac{(1-\kappa)(1-\rho_i)(\cosh \sigma D - 1) + [\beta(1+\rho_i)(1+\kappa) + \kappa \sigma D(1-\rho_i)] \sinh \sigma D}{\text{etc}} \quad (88)$$

$$f_4 \equiv \frac{(1+\kappa)[(1-\rho_i)(\cosh \sigma D - 1) + \beta(1+\rho_i) \sinh \sigma D]}{\text{etc}} \quad (89)$$

$$f_5 \equiv \frac{(1-\rho_i)[-2(\cosh \sigma D - 1) + \sigma D \sinh \sigma D]}{\text{etc}} \quad (90)$$

$$f_6 \equiv \frac{[\sigma D(1+\kappa)(1-\rho_i) + 2\beta\kappa(1+\rho_i)](\cosh \sigma D - 1) + \beta\sigma D(1-\rho_i) \sinh \sigma D}{n^2 \text{ etc}} \quad (91)$$

It is useful to note that,

$$f_1 + f_2 = f_3 + f_4 = 1 \quad (92)$$

$$f_3 - f_4 = \kappa f_5 \quad (93)$$

Then the desired terms are (where  $Q_g$  is the negative of the heat removed by conduction, and therefore  $\eta \equiv b \frac{Q_g}{k}$ ):

$$I_e = f_1 I_i + f_2 J_i + f_5 \kappa \frac{Q_g}{-g} \quad (94)$$

$$J_e = f_2 I_i + f_1 J_i - f_5 \kappa \frac{Q_g}{-g} \quad (95)$$

$$E_{ao} = f_3 I_i + f_4 J_i - f_6 \frac{\kappa}{2\beta} \frac{Q_g}{-g} \quad (96)$$

$$E_{ad} = f_4 I_i + f_3 J_i + f_6 \frac{\kappa}{2\beta} \frac{Q_g}{-g} \quad (97)$$

This gives an idea of how the theory can be used to calculate radiation transfer; the actual choice of parameters is arbitrary. For instance, it would be possible to specify the surface temperatures and perhaps assume no incident fluxes (they might conveniently be chosen as being zero), then solve for the emitted fluxes and the surface gradients necessary to maintain the given situation.

### 3.1.8 Derivation of the Effective Thermal Conductivity of a Powder.

Using the relations derived above, it is possible to calculate an effective thermal conductivity for a powder if it is assumed that the powder consists of layers of material the thickness of which is the same as the average particle size of the particles of the powder and if the porosity of

the model is the same as the porosity of the powder. An amount of radiation  $I_{i1}$  is incident on the first layer of the system; the radiation being emitted then from its second surface is  $I_{e1}$ . This latter amount is now incident on the second layer and is therefore the same as  $I_{i2}$ . On the last layer of the system, the  $j^{\text{th}}$  layer, the amount of radiation incident is  $J_{ij}$  (it is in the backward direction), and this layer emits from its other surface an amount of radiation  $J_{ej}$ , which is then incident on the next to last particle and would then have the notation there of  $J_{i(j-1)}$ . In this manner, all the radiation terms can be found as functions of the radiation incident on the first and last layers. It is assumed that the first layer is the hottest one and heat is transferred down the stack to the  $j^{\text{th}}$  layer.

Since the emitted radiation from one layer is incident on the next, one can transform equations (94) and (95) to the following:

$$I_{en} - f_1 I_{e(n-1)} - f_2 J_{e(n+1)} - f_5 \kappa \frac{Q}{g} = 0 \quad (98)$$

and:

$$J_{en} - f_2 I_{e(n-1)} - f_1 J_{e(n+1)} + f_5 \kappa k_g \frac{Q}{g} = 0 \quad (99)$$

But since  $f_1 = 1 - f_2$ , substituting  $-I_{e(n-1)} + f_2 I_{e(n-1)}$  for  $-f_1 I_{e(n-1)}$  in equation (98) and  $-J_{e(n+1)} + f_2 J_{e(n+1)}$  for  $-f_1 J_{e(n+1)}$  in equation (99), the following is obtained:

$$I_{en} - I_{e(n-1)} + f_2 [I_{e(n-1)} - J_{e(n+1)}] - f_5 \kappa \frac{Q}{g} = 0 \quad (100)$$

and:

$$J_{en} - J_{e(n+1)} - f_2 [I_{e(n-1)} - J_{e(n+1)}] + f_5 \kappa \frac{Q}{g} = 0 \quad (101)$$

Similarly, substituting for  $f_2 J_{e(n+1)}$  in equation (98),  $f_2 I_{e(n-1)}$  in equation (99). The following is obtained:

$$I_{en} - J_{e(n+1)} - f_1 [I_{e(n-1)} - J_{e(n+1)}] + f_5 \kappa \frac{Q}{g} = 0 \quad (102)$$

and:

$$J_{en} - I_{e(n-1)} + f_1 [I_{e(n-1)} - J_{e(n+1)}] - f_5 \kappa \frac{Q}{g} = 0 \quad (103)$$

Applying equation (95) to the (n+1) layer:

$$I_{en} = \frac{1}{f_2} J_{e(n+1)} - \frac{f_1}{f_2} J_{e(n+2)} + \frac{f_5}{f_2} \kappa \frac{Q}{g} \quad (104)$$

and equation (94) to the (n-1) layer:

$$J_{en} = \frac{1}{f_2} I_{e(n-1)} - \frac{f_1}{f_2} J_{e(n-2)} - \frac{f_5}{f_2} \kappa \frac{Q}{g} \quad (105)$$

substituting equation (104) into equation (102):

$$\begin{aligned}
J_{e(n+1)} - f_1 J_{e(n+2)} + f_5 \kappa \frac{Q}{-g} - f_2 J_{e(n+1)} \\
- f_1 f_2 [I_{e(n-1)} - J_{e(n+1)}] - f_2 f_5 \kappa \frac{Q}{-g} = 0
\end{aligned} \quad (106)$$

Or, since  $1-f_2 = f_1$ , and dividing by  $f_1$ :

$$J_{e(n+1)} - J_{e(n+2)} - f_2 [I_{e(n-1)} - J_{e(n+1)}] + f_5 \kappa \frac{Q}{-g} = 0 \quad (107)$$

Similarly, substituting equation (105) into equation (103):

$$I_{e(n-1)} - I_{e(n-2)} + f_2 [I_{e(n-1)} - J_{e(n+1)}] - f_5 \kappa \frac{Q}{-g} = 0 \quad (108)$$

From equations (100), (101), (107), and (108), it can be seen that:

$$\begin{aligned}
J_{en} - J_{e(n+1)} &= J_{e(n+1)} - J_{e(n+2)} = I_{e(n-2)} - I_{e(n-1)} = \\
J_{e(n-1)} - I_{en} &= I_{en} - I_{e(n+1)} = I_{e(n+1)} - I_{e(n+2)}
\end{aligned} \quad (109)$$

Furthermore,

$$\begin{aligned}
I_{e(n-1)} - J_{e(n+1)} &= I_{e(n-2)} - f_2 [I_{e(n-1)} - J_{e(n+1)}] + f_5 \kappa \frac{Q}{-g} \\
&\quad - J_{e(n+1)} - f_2 [I_{e(n-1)} - J_{e(n+1)}] + f_5 \kappa \frac{Q}{-g} \\
&= I_{e(n+2)} - J_{e(n+1)} - 2 f_2 [I_{e(n+1)} - J_{e(n+1)}] \\
&\quad + 2 f_5 \kappa \frac{Q}{-g}
\end{aligned} \tag{110}$$

Or by repeated substitutions similar to equation (109), if there are  $j$  particles,

$$\begin{aligned}
I_{e(n-1)} - J_{e(n+1)} &= I_{i1} - J_{ij} - (j-1) f_2 [I_{e(n-1)} - J_{e(n+1)}] \\
&\quad + (j-1) f_5 \kappa \frac{Q}{-g}
\end{aligned} \tag{111}$$

and:

$$I_{e(n-1)} - J_{e(n+1)} = \frac{I_{i1} - J_{ij} + (j-1) f_5 \kappa \frac{Q}{-g}}{1 + f_2 (j-1)} \tag{112}$$

and:

$$\begin{aligned}
J_{en} - J_{e(n+1)} &= J_{e(n+1)} - J_{e(n+2)} = I_{en} - J_{e(n+1)} = \\
I_{e(n+1)} - I_{e(n+2)} &= \frac{f_2 [I_{i1} - J_{ij}] - f_5 \kappa \frac{Q}{-g}}{1 + f_2 (j-1)}
\end{aligned} \tag{113}$$

Since the amount of heat transferred by gas conduction is the same everywhere in the system, it would simplify matters if the following substitution is made:

$$\frac{(dT)}{(dx)_g} = \frac{-[E_{ad(n-1)} - E_{aon}]}{b_1 D_p} \quad (114)$$

The notation  $b_1$  (which is defined as being equal to  $4 \sigma' T^3$ ) is here used to denote transfer in air where the index of refraction is unity as differentiated from  $b$  (which equals  $4 \sigma' n^2 T^3$ ) which is used for transfer within a layer of index of refraction  $n$ .

Where it should be remembered that:

$$Q_g = -k_g \frac{(dT)}{(dx)_g} \quad (115)$$

$D_p$  represents the distance across the pores and  $\frac{E_{ad(n-1)} - E_{aon}}{b_1}$  is the temperature drop across the pore. The latter can be evaluated from equations (96) and (97):

$$[E_{ad(n-1)} - E_{aon}] = f_4 I_{e(n-2)} + f_3 J_{en} + f_6 \kappa \frac{k_g}{\beta} \frac{(dT)}{(dx)_g} - f_3 I_{e(n-1)} - f_4 J_{e(n+1)} \quad (116)$$

Or, on substituting for  $I_{e(n-2)}$  and  $J_{en}$  from equations (108) and

(101), and using the relationship  $f_3 - f_4 = \kappa f_5$ :

$$[E_{ad(n-1)} - E_{aon}] = \frac{[I_{e(n-1)} - J_{e(n+1)}][f_2 - \kappa f_5]}{1 + \kappa \frac{k_g}{b_{1D_p}} \left[ \frac{f_6}{\beta} - f_5 \right]} \quad (117)$$

Finally, by substituting for  $I_{e(n-1)} - J_{e(n+1)}$  from equation (112), again solving for  $E_{ad(n-1)} - E_{aon}$  and reducing to simplest terms we finally obtain the latter as a function of  $I_{il} J_{ij}$  and  $k_g$ :

$$[E_{ad(n-1)} - E_{aon}] = \frac{[I_{il} - J_{ij}][f_2 - \kappa f_5]}{1 + f_2(j-1) + \frac{\kappa k_g}{b_{1D_p}} \left\{ \frac{f_6[1+f_2(j-1)]}{\beta} - f_5[\kappa f_5(j-1)+1] \right\}} \quad (118)$$

Introducing equation (118) into equation (113), the following is obtained:

$$\begin{aligned} J_{en} - J_{e(n+1)} &= J_{e(n+1)} - J_{e(n+2)} = I_{en} - I_{e(n+1)} = I_{e(n+1)} - I_{e(n+2)} \\ &= \frac{[I_{il} - J_{ij}] \left\{ f_2 + \frac{\kappa k_g}{b_{1D_p}} \left[ \frac{f_2 f_6}{\beta} - \kappa f_5^2 \right] \right\}}{1 + f_2(j-1) + \frac{\kappa k_g}{b_{1D_p}} \left\{ \frac{f_6}{\beta} [1+f_2(j-1)] - f_5[\kappa f_5(j-1)+1] \right\}} \end{aligned} \quad (119)$$

Using the above equation, any desired term can now be derived. For instance:

$$I_{el} = I_{il} - \frac{[I_{il} - J_{ij}]\{f_2 + \frac{\kappa k_g}{b_{1Dp}} [\frac{f_2 f_6}{\beta} - \kappa f_5^2]\}}{1 + f_2(j-1) + \frac{\kappa k_g}{b_{1Dp}} \{ \frac{f_6}{\beta} [1 + f_2(j-1) - f_5[\kappa f_5(j-1) + 1]] \}} \quad (120)$$

and:

$$I_{en} = I_{il} - \frac{n[I_{il} - J_{ij}]\{f_2 + \frac{\kappa k_g}{b_{1Dp}} [\frac{f_2 f_6}{\beta} - \kappa f_5^2]\}}{1 + f_2(j-1) + \frac{\kappa k_g}{b_{1Dp}} \{ \frac{f_6}{\beta} [1 + f_2(j-1) - f_5[\kappa f_5(j-1) + 1]] \}} \quad (121)$$

similarly:

$$J_{en} = J_{ij} + \frac{(j-n+1)[I_{il} - J_{ij}]\{f_2 + \frac{\kappa k_g}{b_{1Dp}} [\frac{f_2 f_6}{\beta} - \kappa f_5^2]\}}{1 + f_2(j-1) + \frac{\kappa k_g}{b_{1Dp}} \{ \frac{f_6}{\beta} [1 + f_2(j-1) - f_5[\kappa f_5(j-1) + 1]] \}} \quad (122)$$

And substituting into equations (96) and (97):

$$E_{aon} = f_3 I_{il} + f_4 J_{ij} + \frac{[I_{il} - J_{ij}]\{[f_4(j-1) - n + 1]\{f_2 + \frac{\kappa k_g}{b_{1Dp}} [\frac{f_2 f_6}{\beta} - \kappa f_5^2]\} + \frac{f_6}{2}[f_2 - \kappa f_5]\}}{1 + f_2(j-1) + \frac{\kappa k_g}{b_{1Dp}} \{ \frac{f_6}{\beta} [1 + f_2(j-1) - f_5(j-1) + 1] \}} \quad (123)$$

and:

$$E_{\text{adn}} = f_4 I_{il} + f_3 J_{ij}$$

$$+ \frac{[I_{il} + J_{ij}][f_3(j-1) - n + 1]\{f_2 + \frac{\kappa k_g}{b_1 D_p} [\frac{f_2 f_6}{\beta} - \kappa f_5^2]\} \frac{f_6}{2\beta} [f_2 - \kappa f_5] \}}{1 + f_2(j-1) + \frac{\kappa k_g}{b_1 D_p} \{ \frac{f_6}{\beta} [1 + f_2(j-1)] - f_5 [\kappa f_5(j-1) + 1] \}}$$

(124)

These relationships allow the calculation of any desired radiation term or temperature in a system if the incident radiation, the conductivity of the gas, the scattering and absorption coefficients, the average temperature, the conductivity of the solid, and the average size and number of particles in the system are known.

An effective thermal conductivity  $k_e$  is defined by the following equation:

$$Q = -k_e \frac{(\Delta T)}{(\Delta x)} \quad (125)$$

or:

$$k_e = -Q \frac{(\Delta x)}{(\Delta T)} \quad (126)$$

where  $Q$  is the heat flowing across a unit area under the influence of a temperature drop,  $\Delta T$ , which occurs over a representative distance,  $\Delta x$ . The length from the top surface of one layer to the top surface of the next layer will be taken as a representative distance. Then:

$$\Delta x = (D + D_p) \quad (127)$$

where  $D$  is the thickness of a layer, and  $D_p$  is the thickness of a pore. The temperature drop across this section is then:

$$\Delta T = \frac{[E_{ad(n-1)} - E_{aon}]}{b_1} \quad (128)$$

which from equation (123) is found to be:

$$\Delta T = \frac{(I_{i1} - J_{ij})}{b_1} \cdot \frac{[f_2 + \frac{\kappa k_g}{b_1 D_p} (\frac{f_2 f_6}{\beta} - \kappa f_5^2)]}{\{1 + f_2(j-1) + \frac{\kappa k_g}{b_1 D_p} [\frac{f_6}{\beta} [1 + f_2(j-1)] - f_5[\kappa f_5(j-1) +]]\}} \quad (129)$$

The total heat flow can be found if any surface is considered. There the heat flow is equal to the difference between the forward and backward radiation fluxes plus the heat conducted by the gas. The latter is equal to the temperature drop across a pore times the thermal conductivity of the gas divided by the distance across the pore or:

$$\begin{aligned} Q &= I_{in} - J_{en} + \frac{k_g}{b_1 D_p} [E_{ad(n-1)} - E_{aon}] \\ &= I_{e(n-1)} - J_{en} + \frac{k_g}{b_1 D_p} [E_{ad(n-1)} - E_{aon}] \end{aligned} \quad (130)$$

Introducing equations (121) through (124) into equation (130) clearing of fractions and reducing to the simplest terms, the following is obtained:

$$Q = \frac{[I_{il} - J_{ij}] \left\{ f_1 + \frac{k_g}{b_1 D_p} \left[ \frac{\kappa f_6 f_1}{\beta} - f_1 + 4f_4^2 \right] \right\}}{1 + f_2(j-1) + \frac{\kappa k_g}{b_1 D_p} \left\{ \frac{f_6}{\beta} [1 + f_2(j-1)] - f_5 [\kappa f_5(j-1) + 1] \right\}} \quad (131)$$

Then from equations (125) to (131) we find:

$$k_e = \frac{b_1(D+L_p) \left\{ f_1 + \frac{k_g}{b_1 D_p} \left[ \frac{\kappa f_6 f_1}{\beta} - f_1 + 4f_4^2 \right] \right\}}{f_2 + \frac{\kappa k_g}{b_1 D_p} \left[ \frac{f_2 f_6}{\beta} - \kappa f_5^2 \right]} \quad (132)$$

Since the pore size of powder is difficult to measure but the particle size and porosity can be measured, it is desirable to substitute for  $D_p$  a function of the porosity,  $P$ , and the particle size,  $D$ :

$$P = \frac{D}{D + D_p} \quad (133)$$

or:

$$D_p = \frac{PD}{(1-P)} \quad (134)$$

and:

$$D_p + D = \frac{D}{(1-P)} \quad (135)$$

Then the conductivity becomes:

$$k_e = \frac{D k \sigma \{b_1 \beta P D f_1 + k_g (1-P) [\kappa f_6 f_1 - \beta f_1 + 4\beta f_4^2]\}}{(1-P) \beta \{P D k \sigma f_2 + 2n^2 k_g (1-P) [f_2 f_6 - \beta f_5^2]\}} \quad (136)$$

Note that the  $\kappa$  in the second part of the denominator has been removed and  $2b\beta/k\sigma$  has been introduced in its place, (where  $k$  is the thermal conductivity of the solid) and that the equation has been cleared of fractions. At this point, to simplify the above expression to one that can be used, it is necessary to make a tedious substitution for  $f_1$  through  $f_6$  from equations (85) through (91). This will be done only for the case of a vacuum; i.e., where  $k_g$  is zero.

The result is:

$$k_e = \frac{2\beta D b_1 (1-\rho_o) (1+\kappa) \sinh \sigma D}{(1-P) \{2(1-\rho_i) (\cosh \sigma D - 1) + [2\beta (\rho_o + \rho_i) (1+\kappa) + \kappa \sigma D (1-\rho_i)] \sinh \sigma D\}} \quad (137)$$

This equation illustrates the usefulness of the general theory in that a relatively simple equation can be derived for a very complex problem. This equation has proper end points, partially due to the introduction of the surface reflections.

### 3.2 Validity of the Hamaker Approximation

In development of the Hamaker equations, an approximation is introduced in the conductive part of the combined heat transfer by conduction and radiation. Measurements of the effective thermal conductivity of polycrystalline alumina at temperatures of about 2000°K indicate that the conductive part is still a major contributor<sup>(14)</sup>. This results from the relatively high conductivity and large scattering in dense polycrystalline ceramics. Since the utility of the specific

solutions of the Hamaker equation derived thus far depends critically on the validity of our underlying assumptions, it is essential to have critical understanding of the nature of these limitations.

### 3.2.1 The Hamaker Approximation

The equations describing the transmission of heat by conduction and radiative transfer in a homogeneous medium are (for the one dimensional case and the steady state):

$$k \frac{d^2 T}{dx^2} - 2an^2 \sigma' T^4 + a(I+J) = 0 \quad (138)$$

$$\frac{dI}{dx} = an^2 \sigma' T^4 - (a+s)I + sJ \quad (139)$$

$$\frac{dJ}{dx} = (a+s)J - sI - an^2 \sigma' T^4 \quad (140)$$

where  $I$  and  $J$  are the radiant fluxes in the  $+x$  and  $-x$  directions,  $T$  is the absolute temperature,  $\sigma'$  is the Boltzmann constant, and  $k, a, s, n$  are material constants already defined. These equations are identical to (51) (55) and (56).

Hamaker's approximation consists of replacing the first term in equation (138),  $k \frac{d^2 T}{dx^2}$ , by  $\frac{k}{T_0} \frac{d^2 (T^4)}{dx^2}$ , in which

$T_0$  is some average temperature in the material. This replacement makes the set of equations (138), (139), (140) linear and homogeneous in the quantities  $I, J$ , and  $T^4$  so that a solution in closed form can be obtained readily. However, consider the identity:

$$k \frac{d^2 T}{dx^2} = - \frac{3k}{T} \left( \frac{dT}{dx} \right)^2 + \frac{k}{4T^3} \frac{d^2(T^4)}{dx^2} \quad (141)$$

Hamaker's approximation is equivalent to neglecting the first term on the right hand side: it should be valid therefore only when the inequality, (5),

$$\left( \frac{dT}{dx} \right)^2 \ll \frac{T}{3} \frac{d^2 T}{dx^2} \quad (142)$$

is satisfied or when the entire conduction term,  $k d^2 T/dx^2$  negligible compared to the other terms in equation (138). We note that the inequality (142) cannot be satisfied with a constant temperature gradient (i.e. when conduction dominates radiative transfer) and, in fact, is not satisfied for any temperature distribution of the form  $T = ax^n$  unless  $n \ll \frac{1}{3}$ .

### 3.2.2 Numerical Integration of Exact Equations at Low Temperatures

For computation it is convenient to reduce the set of equations (138), (139), (140) to a single second order equation. Equations (139) and (140) are added and the sum integrated once to give:

$$I + J = -(a+2s) (Hx+kT) + A \quad (143)$$

in which A is a constant of integration and H is the net heat flux (also constant) defined by:

$$H = I - J - k \frac{dT}{dx} \quad (144)$$

Substituting in equation (138), we obtain finally:

$$k \frac{d^2 T}{dx^2} - 2an^2 \sigma' T^4 - a(a+2s)(Hx+kT) + aA = 0 \quad (145)$$

This equation has been given essentially by Kellett; we have introduced here the effect of radiation scattering and multiplied the Boltzmann constant by  $n^2$ . For comparison with various approximations we have calculated an exact solution to equation (145) by numerical integration for the initial conditions (imposed at  $x=0$ ) listed below:

- a. Heat flow in  $-x$  direction in a homogeneous medium containing the plane  $x=0$
- b. Net flux,  $H = -3.03 \text{ cal. cm}^{-2} \text{ sec}^{-1}$
- c. Radiant flux in  $+x$  direction,  $I(x) = 0$  at  $x = 0$
- d. Radiant flux in  $-x$  direction,  $J(x) = -0.03$  at  $x=0$  ( $\text{cal cm}^{-2} \text{ sec}^{-1}$ )
- e. Temperature,  $T(x) = 0.1$  at  $x=0$  (in units of  $1000^\circ\text{K}$ ).
- f. Temperature gradient,  $\frac{dT}{dx} = 1.0$  at  $x=0$  (units of  $1000^\circ\text{K cm}^{-1}$ )

The material constants assumed are:

- g. Absorption coefficient,  $a = 10 \text{ cm}^{-1}$
- h. Scattering coefficient,  $s = 10 \text{ cm}^{-1}$
- i. Material conductivity,  $k = 3 \text{ cal cm}^{-1} \text{ sec}^{-1} \text{ per } 1000^\circ\text{K}$
- j. Refractive index and radiation constant,  
 $n^2 \sigma' = 3 \text{ cal cm}^{-2} \text{ sec}^{-1} \text{ per } (1000^\circ\text{K})^4$

For the particular values assumed, equation (145) becomes:

$$\frac{d^2T}{dx^2} = 20 T^4 + 300 T - 303 x - 30.1 \quad (146)$$

A straightforward numerical integration gives the solutions plotted in Figure 1. It should be noted that the initial temperature was chosen to be very low and the temperature gradient is large so the solutions obtained thus far cover a range in which substantial differences between the Hamaker and exact solutions are to be expected.

### 3.2.3 Numerical Integration of Exact Equations at High Temperatures

We have also attempted the numerical integration of the transfer equation (145) in the temperature range of 1000°K and above. In principle a numerical solution can be obtained to arbitrarily high accuracy by taking sufficiently small intervals for the integration; in practice a very large number of intervals is needed to get reasonable accuracy in the high temperature region. As the solution proceeds to higher temperatures the second term,  $2 an^2c'T^4$ , dominates; to satisfy the equation the first term,  $k \frac{d^2T}{dx^2}$ , must become large and positive with the result that the solution  $T(x)$  rapidly goes to infinity.

A complete analytical solution for equation (145) has not been found but we have developed a form which has the correct asymptotic behavior and is well adapted to numerical calculation in the high temperature limit. We can show that if  $T(x)$  is a solution of equation (145), then the inverse function  $x(T)$  must satisfy equation (147):

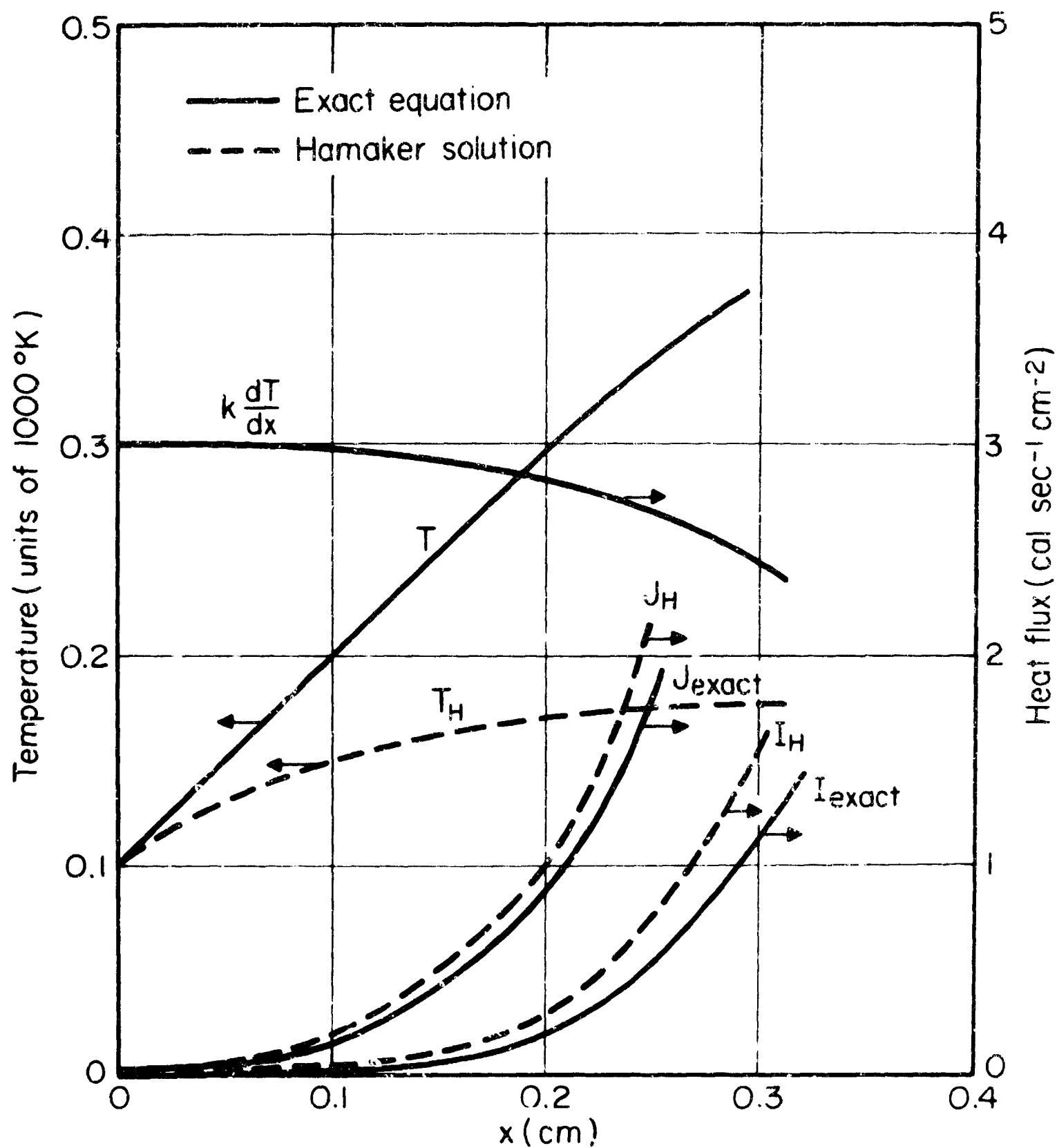


Figure 1: Comparison of numerical integration of exact solution and Hamaker solution for conditions described in text

$$\begin{aligned}
 x(T) = \int_{T_0}^T d\theta \left\{ \frac{4}{5} \frac{an^2\sigma'}{k} (\theta^5 - T_0^5) + a(a+2s)(\theta^2 - T_0^2) - \right. \\
 \left. \frac{2aA}{k} (\theta - T_0) + \left(\frac{dT}{dx_0}\right)^2 + \frac{2a(a+2s)}{k} H \int_{T_0}^{\theta} x(\xi) d\xi \right\} - \frac{1}{2}
 \end{aligned}
 \tag{147}$$

The two variables of integration,  $\theta$  and  $\xi$ , have the dimensions of temperature:  $\theta$  ranges from  $T_0$  to  $T$  and  $\xi$  ranges from  $T_0$  to  $\theta$ .

After differentiating once, equation (147) becomes:

$$\begin{aligned}
 \frac{dx}{dT} = \left\{ \frac{4}{5} \frac{an^2\sigma'}{k} (T^5 - T_0^5) + a(a+2s)(T^2 - T_0^2) - \frac{2aA}{k} (T - T_0) + \right. \\
 \left. \left(\frac{dT}{dx_0}\right)^2 + \frac{2a(a+2s)}{k} H \int_{T_0}^T x(\xi) d\xi \right\} - \frac{1}{2}
 \end{aligned}
 \tag{148}$$

Although formidable at first sight, these equations are not difficult to apply. A procedure for calculating the solution  $x(T)$  follows:

- a. Assume the materials constants, the heat fluxes  $H$  and  $A$ , and the initial conditions  $T_0$  and  $\left(\frac{dT}{dx_0}\right)$  at  $x_0=0$
- b. Estimate the coordinate  $x_1$  corresponding to a nearby temperature  $T_1$  and evaluate the integral  $\int_{T_0}^{T_1} x(\xi) d\xi$ .
- c. Calculate  $\frac{dx}{dT}$  at  $T = T_1$  from equation (148).

- d. Integrate to get  $x_1 = \int_{T_0}^{T_1} \left( \frac{dx}{dT} \right) dT + x_0$ .
- e. Use the calculated value of  $x_1$  to replace the estimated value in (b) and iterate until a consistent value for  $x_1$  is obtained.
- f. Continue stepwise to calculate  $x_2(T_2)$ ,  $x_3(T_3)$  and so forth. As the solution proceeds to high temperature the step (b) and the iteration becomes unnecessary because the term involving  $\int_{T_0}^{T_{n+1}} x(\xi) d\xi$  becomes negligible compared to the first term on the right hand side of equation (148).

As an illustration of the method we have calculated a solution with the same material constants and initial values for  $H$ ,  $I$ ,  $J$ , and  $\left( \frac{dT}{dx_0} \right)$  used in the last section except that the initial temperature is taken to be  $1000^\circ\text{K}$  ( $T_0=1.0$ ). Equation (148) then becomes:

$$\frac{dx}{dT} = \{ 8(T^5-1) + 300(T^2-1) - 60.2(T-1) + 1-606 \int_1^T x d\xi \}^{-\frac{1}{2}} \quad (149)$$

The result is plotted in Figure 2.

To show that equations (145) and (147) are equivalent,

differentiate equation (147) with respect to  $T$  to obtain equation (148) which can be written as:

$$\begin{aligned} \frac{dT}{dx} = \left( \frac{dx}{dT} \right)^{-1} = & \left\{ \frac{4}{5} \frac{an^2 \sigma'}{k} (T^5 - T_0^5) + a(a+2s)(T^2 - T_0^2) - \right. \\ & \left. \frac{2aA}{k} (T - T_0) + \left( \frac{dT}{dx_0} \right)^2 + \frac{2a(a+2s)}{k} H \int_{T_0}^T x(\xi) d\xi \right\}^{-\frac{1}{2}} \quad (150) \end{aligned}$$